

Conjugacy action, induced representations and the Steinberg square for simple groups of Lie type

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ABSTRACT

Let G be a finite simple group of Lie type, and let π_G be the permutation representation of G associated with the action of G on itself by conjugation. We prove that every irreducible representation of G is a constituent of π_G , unless $G = \mathrm{PSU}_n(q)$ and n is coprime to $2(q+1)$, where precisely one irreducible representation fails. Let St be the Steinberg representation of G . We prove that an irreducible representation of G is a constituent of the tensor square $\mathrm{St} \otimes \mathrm{St}$, with the same exceptions as in the previous statement.

1. Introduction

The two main problems discussed in this paper do not appear at first sight to be connected. The first concerns the permutation representation π_G of a finite group G in its action on itself by conjugation (also called the adjoint action). The second is concerned with the irreducible constituents of the tensor square of the Steinberg representation of a simple group of Lie type. In addition, we also deal with the existence of a maximal torus T of G such that every irreducible representation of G is a constituent of the induced representation $\mathrm{Ind}_T^G(1_T)$.

The conjugation action of G on itself, the afforded $\mathbb{C}G$ -module Π_G and its character π_G are standard objects of study in the representation theory of finite groups. For instance, the submodule of G -invariants of this module forms the center of the group algebra $\mathbb{C}G$. The multiplicity of the trivial $\mathbb{C}G$ -module in Π_G equals the number of conjugacy classes in G . Surprisingly, almost nothing is known about the multiplicities of other irreducible modules in Π_G . The problem has a long history; the initial question was to determine those irreducibles of multiplicity zero. See [23], [24], [22] and the bibliography there. Passman restated this problem to that of determining the kernel Δ_G of the representation $\mathbb{C}G \rightarrow \mathrm{End}(\Pi_G)$, and studied the problem both for finite and infinite groups G . It becomes clear that the problem cannot be answered in simple terms for arbitrary finite groups.

Significant progress was achieved in [14], where Passman's problem was solved for finite classical simple groups. For alternating and sporadic simple groups Δ_G was proved to be zero in [15], but a family of unitary groups $\mathrm{PSU}_n(q)$ with n coprime to $2(q+1)$ yields examples with $\Delta_G \neq 0$. (The first example of this, $\mathrm{PSU}_3(3)$, was given by Frame in his review of [23].) For other simple groups the problem remained open, and is finally solved in this paper. In addition, we determine Δ_G in the above exceptional cases.

THEOREM 1.1. *Let G be a finite simple group of Lie type, other than $\mathrm{PSU}_n(q)$ with $n \geq 3$ coprime to $2(q+1)$. Then $\Delta(G) = 0$, that is, every complex irreducible character of G is a constituent of the conjugation permutation character of G .*

In the exceptional case, $\Delta(G) \cong \text{Mat}_m(\mathbb{C})$, where $m = (q^n - q)/(q + 1)$, that is, every complex irreducible character occurs, except precisely the (unique) irreducible character of dimension $(q^n - q)/(q + 1)$.

The exception is rather interesting: the unique missing irreducible character is the irreducible of smallest dimension greater than 1, one of the Weil modules.

The second problem we address in this paper concerns the square of the Steinberg character of a simple group of Lie type. The initial question was partly motivated by a conjecture of John Thompson, that each finite simple group possesses a conjugacy class the square of which covers the group. The natural question then is whether each finite simple group has an irreducible character whose tensor square contains each irreducible of the group. The answer turns out to be negative in general, see Lemma 5.3. However, we show that the answer is positive for almost every group, with the same exceptions as in Theorem 1.1:

THEOREM 1.2. *Let G be a finite simple group of Lie type, other than $PSU_n(q)$ with $n \geq 3$ coprime to $2(q + 1)$. Then every irreducible character of G is a constituent of the tensor square St^2 of the Steinberg character St of G .*

In the exceptional case, every irreducible character occurs, except precisely the (unique) irreducible character of dimension $(q^n - q)/(q + 1)$.

REMARK 1.3. 1) The assertion also holds for the almost simple but imperfect groups $Sp_4(2)$, ${}^2G_2(3)$, ${}^2F_4(2)$, and $G_2(2)$. Moreover, in the first three of these, we can take the Steinberg character of the derived subgroup to be any irreducible constituent of the restriction of Steinberg character of the group; the assertion remains valid in the simple group.

2) In the exceptional case of Theorem 1.2, the square of no irreducible character contains every irreducible of G as a constituent, cf. Lemma 5.3. In other words, in case of groups of Lie type, the best possible result for the tensor square conjecture is achieved by our choice of Steinberg module. This also implies that the assertion fails for the group $G_2(2)'$, in view of its isomorphism to $PSU_3(3)$.

3) Eamonn O'Brien has checked for us that each sporadic simple group G possesses an irreducible character whose square involves each irreducible character of G as a constituent, and the same holds also for the alternating groups of degree at most 17. We believe that this is true for the alternating groups in general.

We obtain two corollaries at this stage.

According to an observation of L. Solomon [25], the multiplicity of an irreducible character in the conjugation character of G equals the sum of entries in the corresponding row of the character table of G . Hence we have:

COROLLARY 1.4. *Let G be a non-abelian finite simple group, other than $PSU_n(q)$ with $n \geq 3$ coprime to $2(q + 1)$. The sum of entries in each row of the character table of G is a positive integer.*

In the exceptional case, the same is true of each row except for the second row.

COROLLARY 1.5. *Let \mathbf{G} be a simple simply connected algebraic group, and let Fr denote a Frobenius endomorphism of \mathbf{G} . Set $G = \mathbf{G}^{Fr}$ and $H = \mathbf{G}^{Fr^2}$. Then the restriction $\text{St}^H|_G$ of*

the Steinberg representation St^H to G equals $\text{St}^G \otimes \text{St}^G$, and hence its irreducible constituents are as described in Theorem 1.2.

Indeed, it was proved by Steinberg [28, statement (3), p. 101] that the restriction of St^H to G coincides with the square of St^G . So the result follows from Theorem 1.2.

It follows from the above theorems that for simple groups of Lie type the Steinberg square character problem has exactly the same answer as Passman's problem discussed above. The cause of this coincidence will be seen in Section 5. In one direction, this is clear: observe that $\Pi_G = \sum_{M \in \text{Irr } G} M \otimes M^*$. Consider the module $\Phi_G = \sum_{M \in \text{Irr}(G)} M \otimes M = \Phi_1 \oplus \Phi_2$, where $\Phi_1 = \sum_{M \cong M^*} M \otimes M$ and $\Phi_2 = \sum_{M \not\cong M^*} M \otimes M$. Then Φ_1 is isomorphic to a submodule of Π_G , whereas Φ_2 does not contain 1_G as a constituent. Therefore, if $\Delta_G \neq 0$ then the tensor square problem has a negative answer. (This does not require G to be simple.) The converse is more mysterious.

It was conjectured in [15] that, for every simple group G , $\Delta_G = 0$ if and only if there exists a single conjugacy class C such that the permutation module Π_C associated with the action of G on C contains every irreducible representation of G as a constituent. The conjecture was proved to be true in [15] for alternating and sporadic simple groups. In this paper we confirm this conjecture in general, and moreover we prove a more general result:

THEOREM 1.6. *For every simple group G there exists a conjugacy class C such that $\Delta_G = \Delta_C$. Furthermore, if G is a group of Lie type then C can be chosen semisimple.*

Theorem 1.6 already implies the Steinberg square result; indeed, it follows from Deligne and Lusztig [3, 7.15.2] (a result which was suggested by earlier work of Bhama Srinivasan) that ϕ is a constituent of St^2 if and only if $\phi|_T$ contains 1_T for some maximal torus T of G , see Corollary 5.2.

Let G be a quasi-simple groups such that $G/Z(G)$ is a group of Lie type in defining characteristic p . For $\zeta \in \text{Irr}(Z(G))$ we set $\text{Irr}_\zeta G = \{\phi \in \text{Irr}(G) : \phi|_{Z(G)} = \zeta \cdot \text{Id}\}$. Recall that the set $\text{Irr } G$ is the union of p -blocks (in the sense of R. Brauer) and the blocks of positive defect are in bijection with $\text{Irr}(Z(G))$. So $\text{Irr}_\zeta G$ is a block, unless $\zeta = 1_{Z(G)}$; in this case, $\text{Irr}_\zeta G$ consists of two blocks, the second one is of defect 0. Then Theorem 1.6 can be generalized as follows:

THEOREM 1.7. *Let G be a quasi-simple group such that $G/Z(G)$ is a simple group of Lie type in defining characteristic p with $(|Z(G)|, p) = 1$, and let $\zeta \in \text{Irr}(Z(G))$. Then there exists a maximal torus T of G such that $\text{Irr}_\zeta G$ coincides with the set of all irreducible constituents of $\text{Ind}_T^G(\tau)$ for any $\tau \in \text{Irr}(T)$ above ζ , unless $G = \text{SU}_n(q)$, $(2(q+1), n) = 1$.*

In the exceptional case $\text{Ind}_T^G(\tau)$ may not contain a single nontrivial irreducible representations of G which has degree $(q^n - q)/(q+1)$ or $(q^n + 1)/(q+1)$, and this happens precisely for $q^2 - 1$ irreducible representations τ of T . In particular, the only irreducible character of G which is not a constituent of $\text{Ind}_T^G(1_T)$ is the (unique) unipotent character of degree $(q^n - q)/(q+1)$.

In fact, the torus T can be chosen so that the group $T/Z(G)$ is cyclic, and in Theorem 1.6 $C_G(c)$ is cyclic for every c in the class C . By Frobenius reciprocity, ϕ is a constituent of $\text{Ind}_T^G(\tau)$ if and only if $\phi|_T$ contains τ . Thus, Theorem 1.7 is equivalent to the statement that there is a maximal torus T such that, for every non-trivial irreducible representation φ , the restriction $\varphi|_T$ contains all $\tau \in \text{Irr}(T)$ satisfying $\tau|_{Z(G)} = \zeta$. In this form, the result probably

remains valid for cross-characteristic representations too, and we prove this for classical groups (see Theorem 3.1). This is partially based on a result concerning the action of some parabolic subgroup on its unipotent radical:

PROPOSITION 1.8. *Let G be a quasi-simple classical group over a field \mathbb{F}_q of characteristic p . Suppose that G is none of the following groups: $Sp_4(3)$ for $q = 3$, $PSU_n(q)$, n odd and divisible by $q + 1$, and $\Omega_{2n+1}(q)$, $(q - 1)n/2$ even. Then there exists a p -group A and a cyclic self-normalizing subgroup $C \leq N_G(A)$ such that every $N_G(A)$ -orbit on $A \setminus \{1\}$ contains a regular C -orbit.*

In Proposition 1.8 the orbits in question are considered with respect to the conjugation action, and a C -orbit is called regular if its size equals $|C|$.

In the remainder of the paper we prove the above results. Section 2 contains a number of preliminary observations. In Section 3 we prove Theorem 1.7 for classical groups. This part is partially based on the thesis [14] of the first author, written under the supervision of the fourth author. In Section 4, the exceptional groups of Lie type are considered.

Our method is different for classical and exceptional groups of Lie type. In the former case a key tool is in the study of the “internal permutation module” for the conjugacy action of a parabolic subgroup on its unipotent radical. In the latter case, our arguments rely on certain character estimates.

Notation. The greatest common divisor of integers m, n is denoted by (m, n) . Let \mathbb{F}_q be the field of q elements, where q is a power of a prime p . If \mathbb{F} is a field, \mathbb{F}^\times denotes the multiplicative group of \mathbb{F} . The identity $n \times n$ -matrix is denoted by Id . The block-diagonal matrix with blocks A, B is denoted by $\text{diag}(A, B)$, and similarly for more diagonal blocks.

If V is an orthogonal, symplectic or unitary space then $G(V)$ is the group of all linear transformation preserving the unitary, symplectic or orthogonal structure on V . Our notation for classical groups is standard; in particular, if V is orthogonal space then $\Omega(V)$ is the subgroup of $SO(V)$ of elements of spinor norm 1. In the unitary case we denote by σ the automorphism of $GL(V)$ extending the involutory automorphism of the ground field, in other cases, for uniformity, σ is assumed to be the identity automorphism of $GL(V)$.

For a group G we denote by $Z(G)$ the center of G , and for a subset X of G we write $C_G(X)$ for the centralizer and $N_G(X)$ for the normaliser of X in G . We use the symbol 1_G to denote the trivial representation of G of degree 1 or the trivial one-dimensional $\mathbb{F}G$ -module. For a subgroup H of G and an $\mathbb{F}G$ -module M (resp., a representation ϕ) we write $M|_H$ (resp. $\phi|_H$) for the restriction of M (resp., ϕ) to H . To say that M (resp., ϕ) is an irreducible $\mathbb{F}G$ -module (resp., irreducible \mathbb{F} -representation of G) we write $M \in \text{Irr}_{\mathbb{F}}(G)$ (resp., $\phi \in \text{Irr}_{\mathbb{F}}(G)$). If $\mathbb{F} = \mathbb{C}$, the complex number field, we usually drop the subscript \mathbb{C} . If M is an irreducible $\mathbb{F}G$ -module, then $Z(G)$ acts as scalars on M . By a *central character* of M we mean the linear character of $Z(G)$ obtained from this scalar action of $Z(G)$ on M .

2. Preliminaries

Let G be a finite group, let π be the permutation character of G in its conjugation action on itself. The orbits are the conjugacy classes of G . It follows that π is the sum of permutation characters of G on its conjugacy classes. Using Frobenius reciprocity, we have the following lemma (cf. Problem 1.1 of [15]):

LEMMA 2.1. *An irreducible character χ of G is a constituent of π if and only if for some $g \in G$, the restriction of χ to $C_G(g)$ contains the principal character of $C_G(g)$.*

Hence to prove Theorem 1.1 for classical groups, it is sufficient to prove:

THEOREM 2.2. *Let G be a finite simple classical group, other than $PSU_n(q)$ with $n \geq 3$ coprime to $2(q+1)$. Then there exists a cyclic self-centralizing subgroup T of G such that, for every irreducible representation ϕ of G , the trivial representation 1_T is a constituent of the restriction $\phi|_T$, with the exception described in Theorem 1.1. In the exceptional case, the same is true for all but one ϕ , which is the irreducible Weil representation of dimension $(q^n - q)/(q+1)$.*

The exceptional cases were discussed in the earlier paper [15] of two of the authors, and are discussed further in Section 3.5 below.

LEMMA 2.3. *Let G be a finite group with subgroups A, B such that A is abelian and $B \leq N_G(A)$. Let \mathbb{F} be an algebraically closed field of characteristic ℓ , where either $\ell = 0$ or $\ell > 0$ is coprime to $|A|$, and let M be an $\mathbb{F}G$ -module. For an irreducible \mathbb{F} -representation α of A set $M_\alpha := \{m \in M \mid am = \alpha(a)m \text{ for all } a \in A\}$.*

Suppose that $M_\alpha \neq 0$, and moreover the group $C := \text{Stab}_B(\alpha)$ has a common eigenvector $v \in M_\alpha$, that is, $cv = \gamma(c)v$ for all $c \in C$ and some linear \mathbb{F} -character γ of C .

Then $M|_B$ contains the induced module $\text{Ind}_C^B(\gamma)$. In particular, if $C = 1$, then $M|_B$ contains a copy of the regular $\mathbb{F}B$ -module.

Proof. By Maschke's theorem, $M = \bigoplus_\alpha M_\alpha$, and, since $B \leq N_G(A)$, B permutes the summands of this decomposition in accordance to its action on the set of all irreducible \mathbb{F} -representations of A . Decompose $B = \bigcup_{i=1}^n b_i C$ as a disjoint union of C -cosets (with $b_1 = 1$), and set $v_i = b_i v$. Then the vectors v_i belong to pairwise distinct A -eigenspaces $M_{b_i(\alpha)}$, in particular, they are linearly independent. Now B acts on $N := \bigoplus_{i=1}^n \mathbb{F}v_i$, and the B -module N is isomorphic to $\text{Ind}_C^B(\gamma)$. \square

Let r be a p -power, and let σ_0 be either trivial automorphism of \mathbb{F}_r or $r = q^2$ and σ_0 denote the automorphism of \mathbb{F}_r of order 2. We denote by σ the automorphism induced by σ_0 on $GL_n(r)$ and on the algebra of $n \times n$ -matrices over \mathbb{F}_r for any integer $n > 0$.

LEMMA 2.4. (i) *Let $G = GL_n(q)$ and let $s \in G$ be an irreducible element of order $q^n - 1$. Then s is conjugate to s^{-1} in G if and only if $(n, q) \in \{(2, 2), (1, 2), (1, 3)\}$.*

(ii) *s^2 is conjugate to s^{-2} in G if and only if $n \leq 2$ and $q \leq 3$.*

(iii) *Let $r = q^2$, $G = GL_n(r)$, and let $s \in G$ be an irreducible element of order $r^n - 1$. Let σ_0 be the Galois automorphism of \mathbb{F}_r/F_q . Let σ be the automorphism of G induced by σ in the natural way. Set $s_1 = s^{q+1}$. Then s_1 is conjugate to $\sigma(s_1^{-1})$ if and only if $n = 1$ and $q \leq 3$.*

Proof. (i) s is irreducible in $GL_n(q)$ (otherwise $|s|$ divides $q^i - 1$ for some $i < n$). By Schur's lemma, the enveloping \mathbb{F}_q -algebra K of $\langle s \rangle$ is a field. Let V be the natural module for G . Then V is an irreducible K -module, hence $|V| = |K| = q^n$ and s is a generator of the multiplicative group K^\times of K .

If $n = 1$, the claim is trivial, so we assume $n > 1$, hence also $|s| > 2$. Suppose that $s^{-1} = gsg^{-1}$ for some $g \in G$. Then $gKg^{-1} = K$, so g induces a field automorphism on K of order 2 (as $s \neq s^{-1}$). Let $L = C_K(g)$. By Galois theory, $K : L = 2$, and hence $|L| = q^{n/2}$. Then $|K^\times/L^\times| = q^{n/2} + 1$ and $s^{q^{n/2}+1} = t$ is a generator of L^\times . On one hand, $gt = tg$ as $t \in L$. On the other hand, $gtg^{-1} = t^{-1}$ as t is a power of s . Therefore, $t = t^{-1}$ and $t^2 = 1$. It follows that $|L| = 2$ or 3. As $\mathbb{F}_q \cdot \text{Id} \subseteq L$, one observes that $\mathbb{F}_q = L$ so $q = 2$ or 3 and $|V| = |K| = 4$ or 9. Therefore, $n = 2$, and the lemma follows.

(ii) s^2 is irreducible unless s^2 belongs to a proper subfield of F_{q^n} , which is not the case. So $\langle s^2 \rangle = K$. Then, as above, $s^{-2} = gs^2g^{-1}$ implies $K : L = 2$, where $L = C_K(g)$. Then the generator t of L^\times is a power of s^2 , so again $|K| = 4$ or 9.

(iii) It is easy to check that $s_1^2 \neq 1$. As above, s is irreducible in $GL_n(r)$, and K , the enveloping algebra of $\langle s \rangle$ over F_r is a field. Suppose that $\sigma(s_1^{-1}) = gs_1g^{-1}$ for some $g \in G$ and let τ be the automorphism of G defined by $\tau(x) = \sigma(gxg^{-1})$. Then $\tau(s_1) = s_1^{-1}$ hence $\tau(K) = K$ and $\tau^2|_K = \text{Id}$. Let $L = \{a \in K : \tau(a) = a\}$. Then $K : L = 2$. Let $t \in L$ be the generator of L . Then $t = \tau(t) = t^{-1}$ as t is a power of s (because $s_1 \notin L$). Hence $t^2 = 1$ and $|L| \leq 3$. As $\mathbb{F}_q \cdot \text{Id} \subseteq L$, we observe that $\mathbb{F}_q \cong L$, so $q = 2$ or 3. As $\mathbb{F}_r : \mathbb{F}_q = 2 = K : L$ and $\mathbb{F}_q \cong L$, we observe that $K = \mathbb{F}_r$, and hence $n = 1$. The converse is obvious. \square

Let q, r, σ be as above. Let $M = \text{Mat}_n(\mathbb{F}_r)$ be the matrix algebra, and $G = GL_n(r)$. For $x \in G$ consider the mapping $M \rightarrow M$ defined as $m \mapsto xm\sigma(x^t)^{-1}$ for $m \in M$. This mapping is linear, and hence M becomes an $\mathbb{F}_q G$ -module. Let $\varepsilon = \pm 1$, and assume $\varepsilon = 1$ if $\sigma \neq 1$ or $p = 2$. Set

$$L = \{x \in M : \sigma(x^t) = -\varepsilon x\}, \text{ if } p \neq 2 \text{ or } \sigma \neq 1, \quad (2.1)$$

$$L_0 = \{x \in M : x^t = x \text{ and } x \text{ has zero diagonal}\}, \text{ if } p = 2 \text{ and } \sigma = 1. \quad (2.2)$$

Observe that the subspaces L and L_0 are G -stable. The action of G on $\text{Mat}_n(\mathbb{F}_r)$, as well as on L and L_0 , is referred below as the congruence action. This induces the dual action on the character groups $\text{Irr}(L)$. In particular, L and $\text{Irr}(L^+)$ are dual $\mathbb{F}_r G$ -modules, as well as L_0 and $\text{Irr}(L_0^+)$, where L^+, L_0^+ are the additive groups of L, L_0 , respectively. Let V be the natural module for $G = GL_n(r)$. If $p = 2$ and $\sigma = 1$ then L_0 is isomorphic to the module of the alternating forms on V , and the dual of L_0 is isomorphic to the module of the alternating forms on the dual of V (cf. [31, Lemma 2.14]). In addition, the dual of L is isomorphic to the module of the quadratic forms on V (cf. [31, Lemma 2.16]).

LEMMA 2.5. *The kernel of the congruence action of $G = GL_n(r)$ on L coincides with $D_\sigma = \{g \in Z(G) : \sigma(g)g = 1\}$. If $n > 2$ then D_σ is also the kernel of the action of G on L_0 . In addition, $D_\sigma = U_1(r) \cdot \text{Id}$ if V is unitary, otherwise $D_\sigma = \{\lambda \cdot \text{Id} : \lambda \in \mathbb{F}_q, \lambda^2 = 1\}$.*

Proof. This is straightforward. \square

LEMMA 2.6. *Let $G' = SL_n(r)$ and L, L_0 as above. Then G' fixes no non-zero element of L and of its dual module $\text{Irr}(L)$, unless $n = 2$ and $\sigma = 1$, or $n = 1$. The same is true for L_0 and $\text{Irr}(L_0)$. In addition, a similar statement is true for a subgroup $X < SL_4(2)$ isomorphic to the alternating group A_7 .*

Proof. The claim for L is [7, Lemma 3.4]. If L is irreducible then so is the dual $\text{Irr}(L)$, which implies the claim for $\text{Irr}(L)$, and similarly, for $\text{Irr}(L_0)$. Similar argument is valid for L_0 . For the additional statement, if $X \cong \mathbf{A}_7$ fixes an alternating form f then f is non-degenerate as both $V \cong \mathbb{F}_2^4$ and its dual are irreducible $\mathbb{F}_2 X$ -modules. Then X is isomorphic to a subgroup of $Sp_4(2)$, which is false. \square

LEMMA 2.7. [27, Theorem 1.1] *Let $PSL_n(r) \leq X \leq PGL_n(r)$ where $(n, r) \neq (2, 2), (2, 3)$ and let C be a cyclic subgroup of X . Let Π be a permutation X -set, on which $PSL_n(q)$ acts non-trivially. Then one of the following holds:*

- (i) C has a regular orbit on Π ;
- (ii) $X = SL_4(2) \cong \mathbf{A}_8$, $|C| = 15$ or 6 and $|\Pi| = 8$;
- (iii) $X = PGL_2(5) \cong \mathbf{S}_5$, $|C| = 6$ and $|\Pi| = 5$.

We remark that the case $|C| = 6$ in (ii) is missing in the original statement in [27, Theorem 1.1(case b)]. This omission has no effect on any other result in [27].

In what follows, we define a Singer subgroup of $SL_n(r)$ to be an irreducible subgroup of order $(r^n - 1)/(r - 1)$.

LEMMA 2.8. *Let $G' = SL_n(r) \leq G \leq GL_n(r)$, $n > 1$, let S be a Singer subgroup in G' , and L the $\mathbb{F}_r G$ -module defined above. Let α be a non-trivial character of the additive group of L (resp., L_0). Then there exists $\beta \in G'\alpha$ such that $C_S(\beta) = D_\sigma$ except for the case where $n = 2$ and $\sigma = 1$.*

Proof. In the action of G on $\text{Irr}(L)$ an element $g \in G$ sends $\alpha \in \text{Irr}(L)$ to $g(\alpha)$, where $g(\alpha)(l) := \alpha(glg^{-1})$ for all $l \in L$. By Lemma 2.6, G' fixes no non-trivial character $\alpha \in \text{Irr}(L)$, except for the case with $n = 2$, $\sigma = 1$.

In the non-exceptional case $G'\alpha \neq \alpha$. Set $G_0 = \cap_{\gamma \in G\alpha} C_G(\gamma)$. Then G_0 is normal in G , and hence $G_0 < Z(G)$ unless $(n, r) = (2, 2)$ or $(2, 3)$. By Lemma 2.7, there exists $\beta \in G\alpha$ such that for $t \in S$ either $t \in G_0$ or $t\beta \neq \beta$, unless possibly $n = 4, r = 2$ or $n = 2, r = 5$. The latter case appears in the conclusion of the lemma. In the former case, $G' = G = SL_4(2)$, the orbit $G\alpha$ is of size 8 and $C_G(\alpha) \cong \mathbf{A}_7$. In this case the result follows from Lemma 2.6.

Observe that $s\beta = \beta$ for $s \in Z(G)$ implies $s \in D_\sigma \cap G$. Indeed, let $s = \lambda \cdot \text{Id}$. Then $sl\sigma(s) = \lambda\sigma(\lambda)l$, and then $\alpha(sl\sigma(s)) = \lambda\sigma(\lambda)\alpha(l)$ for all $l \in L$. Therefore, $s\alpha = \lambda\sigma(\lambda)\alpha$. It follows that $s\alpha = \alpha$ implies $\lambda\sigma(\lambda) = 1$, and the claim follows from Lemma 2.5.

The lemma follows from this observation, including the case $n = 1$. \square

The following observation is a slight modification of [15, Theorem 1.7], where it is assumed that $Z(G) = 1$.

LEMMA 2.9. *Let G be a finite group with cyclic Sylow p -subgroup P . Suppose that $(p, |Z(G)|) = 1$ and $C_G(g) = Z(G)P$ for $g \in P$ of order p . Let ϕ be an irreducible representation of G over the complex numbers, and let $\phi|_{Z(G)} = \zeta \cdot \text{Id}$, where $\zeta \in \text{Irr}(Z(G))$. Then either $\dim \phi < |P|$ or ϕ is a constituent of $(\zeta \times \lambda_P)^G$ for every $\lambda \in \text{Irr}(P)$.*

Proof. Assume that $\dim \phi \geq |P|$. By [15, Lemma 3.1], $\phi|_P$ contains every irreducible representation of P as a constituent. Therefore, $\phi|_{PZ(G)}$ contains every irreducible representation λ of $PZ(G)$ such that $\lambda|_{Z(G)} = \zeta$. So the result follows by Frobenius reciprocity. \square

3. Classical groups

In this section q is a power of a prime p and \mathbb{F} is an algebraically closed field of characteristic $\ell \neq p$. We denote by V a finite-dimensional vector space over a finite field, endowed by a structure of a non-degenerate unitary, symplectic or orthogonal space, and we denote by $G(V)$ the group of all elements of $GL(V)$ preserving the structure. In this section we prove

THEOREM 3.1. *Let G be a quasi-simple group such that $G/Z(G)$ is a simple classical group in defining characteristic p and $(|Z(G)|, p) = 1$. Then there exists a maximal cyclic torus T of G such that whenever ϕ is a non-trivial irreducible \mathbb{F} -representation of G with central character ζ , every $\tau \in \text{Irr}_{\mathbb{F}} T$ is a constituent of $\phi|_T$, unless $G = SU_n(q)$, $(2(q+1), n) = 1$, and $\zeta = 1_{Z(G)}$ and $\phi = \phi_{\min}$, where ϕ_{\min} the unique irreducible representation of G of degree $(q^n - q)/(q+1)$.*

If G in Theorem 3.1 is simple then there exists a cyclic self-centralizing subgroup C of G such that $\phi|_C = \rho_C^{reg} + \xi$ for some proper representation ξ of C , where ρ_C^{reg} denotes the regular representation of C .

Throughout this section G satisfies the hypothesis of Theorem 3.1, and in every subsection we specify the group $G/Z(G)$ to be considered.

3.1. The groups G with $G/Z(G) = P\Omega_{2n}^-(q)$, $n > 3$, or $PSL_n(q)$, $n > 1$, $(n, q) \neq (2, 2), (2, 3)$

In this section $G/Z(G)$ is one of the above groups, so $G/Z(G)$ is simple. These groups are easier to handle than the remaining ones. Our reasoning is practically identical for both of them, but the orthogonal group case requires more attention to detail.

Let V be an orthogonal space of dimension $2n$ over \mathbb{F}_q of Witt index $n-1$. Let W be a 1-dimensional singular subspace of V . Then we choose a basis b_1, \dots, b_{2n} of V such that $b_1 \in W$, $b_2, \dots, b_{2n-1} \in W^\perp$, and the Gram matrix of V is of shape $\begin{pmatrix} 0 & 0 & 1 \\ 0 & \Gamma & 0 \\ 1 & 0 & 0 \end{pmatrix}$, where Γ is a symmetric matrix of size $2n-2$. (If $p=2$, the Gram matrix is obtained from the corresponding symplectic structure on V .) Set $W_0 = \langle b_2, \dots, b_{2n-1} \rangle_{\mathbb{F}_q}$. Then W_0 is a non-degenerate orthogonal space of dimension $2n-2$ and of Witt index $n-2$.

Set

$$G_1 = SO_{2n}^-(q), \quad H := \{g \in G_1 \mid g \text{ stabilizes } W, W_0, \text{ and } \langle b_{2n} \rangle_{\mathbb{F}_q}\}$$

and

$$A := \{g \in G_1 \mid gb_1 = b_1, \text{ and } g \text{ acts trivially on } W^\perp/W\},$$

It is well known that A is an abelian group and $A < \Omega_{2n}^-(q)$.

One observes that, with respect to the above basis, H and A can be described, respectively, as the sets of matrices

$$\begin{pmatrix} \lambda^{-1} & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & \lambda \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & \Gamma(w)^t & 0 \\ 0 & \text{Id}_{2n-2} & w \\ 0 & 0 & 1 \end{pmatrix},$$

where $w \in W_0$, $\lambda \in F_q^\times$ and $h \in SO(W_0)$. The action of H on A can be described in terms of W_0 as follows. Let $g = \text{diag}(\lambda, h, \lambda^{-1}) \in H$ and let w corresponds to $a \in A$ as above. Then gag^{-1} corresponds to $\lambda h(w)$. In particular, if $gag^{-1} = a$ then $h(w) = \lambda^{-1}w$.

It is well known that $SO(W_0) \cong SO_{2n-2}^-(q)$ contains an irreducible cyclic subgroup S of order $q^{n-1} + 1$, called a Singer subgroup in [17]. Let T_1 be a subgroup of H consisting of matrices $\text{diag}(\lambda, h, \lambda^{-1})$ with $h \in S$. Note that $Z(G_1) < T_1$. If q is even then $(q-1, q^{n-1}+1) = 1$, and

hence $T_1 \cong T_1/Z(G_1)$ is cyclic. We observe that $T_1/Z(G_1)$ is cyclic for q odd too. Indeed, as $(|\mathbb{F}_q^\times|, |S|) = (q-1, q^{n-1}+1) = 2$, it follows that the projection of T_1 into $G_1/\{\pm \text{Id}_{2n}\}$ is a cyclic group.

If $G_1 = SL_n(q)$ then we use V to denote the natural module, and let P be the stabilizer of an $(n-1)$ -dimensional subspace W_0 of V . Define H, A as the subgroup of P formed, respectively, by the matrices of the shape:

$$\begin{pmatrix} h & 0 \\ 0 & \det(h^{-1}) \end{pmatrix} \text{ and } \begin{pmatrix} \text{Id}_{n-1} & w \\ 0 & 1 \end{pmatrix},$$

where $w \in W_0$ and $h \in GL_{n-1}(q)$. Then $A = O_p(P)$ and $P = HA$. Obviously, $H \cong GL_{n-1}(q)$. Let T_1 be a subgroup of H of order $q^{n-1} - 1$, which corresponds to a Singer subgroup S under an isomorphism $H \rightarrow GL_{n-1}(q)$. Again, $Z(G_1) < T_1$.

LEMMA 3.2. *For every $1 \neq a \in A$, $C_{T_1}(a)$ consists of scalar matrices. In other words, $A \setminus \{1\}$ is a union of regular orbits for the quotient group $T_1/Z(G_1)$.*

Proof. Suppose the contrary. Let $g \in C_{T_1}(a)$, $a \neq 1$. Note that S is irreducible on W_0 , and hence is contained in a Singer subgroup S' , say, of $GL(W_0)$. It is well known that if $s \in S$ has an eigenvector on W_0 then s is scalar. We have to show that g is scalar.

Suppose $G_1 = SL_n(q)$. Then $g \in C_{T_1}(a)$ implies $(\det h) \cdot h \cdot w = w$ for some $w \neq 0$. This means that w is an eigenvector for h with eigenvalue $\det h^{-1}$. By the above, h is scalar, so $h = \det h^{-1} \cdot \text{Id}_{n-1}$, and hence $g = \det h^{-1} \cdot \text{Id}_n$ is scalar.

Suppose $G_1 = SO_{2n}^-(q)$. Similarly, $g \in C_{T_1}(a)$ implies $\lambda^{-1} \cdot h \cdot w = w$ with $w \neq 0$. Therefore, h is scalar, and hence $h = \lambda \cdot \text{Id}_{2n-2}$. As $h \in SO(W_0)$, we have $\lambda = \pm 1$, which implies that g is scalar.

It follows that the conjugation action of T_1 on A partitions $A \setminus \{1\}$ as a union of regular orbits for the group $T_1/Z(G_1)$. \square

If $G/Z(G) \cong PSL_n(q)$ then $G/Z(G) \cong G_1/Z(G_1)$. If $G/Z(G) \cong P\Omega_{2n}^-(q)$ then $G/Z(G)$ is isomorphic to a subgroup of index at most 2 in $G_1/Z(G_1)$. Set $\overline{T} = T_1/Z(G_1)$ in the former case, and $\overline{T} = (T_1/Z(G_1)) \cap (G/Z(G))$ in the latter case. As $Z(G_1)$ is the kernel of the action of T_1 on A via conjugation, every $T_1/Z(G_1)$ -orbit on $A \setminus \{1\}$ is regular. Therefore, the same is true for every subgroup of $T_1/Z(G_1)$, in particular for \overline{T} .

Recall that the central character of M is the linear character of $Z(G)$ obtained from the scalar action of $Z(G)$ on M .

PROPOSITION 3.3. *Let $G/Z(G)$ be $PSL_n(q)$, $(n, q) \neq (2, 2), (2, 3)$, or $P\Omega_{2n}^-(q)$, $n > 3$. Let M be a non-trivial irreducible $\mathbb{F}G$ -module. Let T be the preimage in G of the group \overline{T} , and let ζ be the central character of M . Then $M|_T$ contains the induced module $\text{Ind}_{T \cap Z(G)}^T(\zeta)$. In particular, $M|_T$ contains 1_T if $\zeta = 1_{Z(G)}$.*

Proof. Let A be as in Lemma 3.2. As $|Z(G)|$ is coprime to p , G contains a T -invariant subgroup $B \cong A$ such that the conjugation action of T on B is the same as T_1 on A , that is, A, B are isomorphic \overline{T} -sets. Then $B \setminus \{1\}$ is a union of regular $T/(T \cap Z(G))$ -orbits (due to Lemma 3.2). Since the actions of $T/(T \cap Z(G))$ on $B \setminus \{1\}$ and on $\text{Irr}(B) \setminus \{1_B\}$ are dual to each other, we see that the set $\text{Irr}(B) \setminus \{1_B\}$ is a union of regular $T/Z(G)$ -orbits. Then apply Lemma 2.3. \square

3.2. Symplectic groups

For symplectic groups $G = Sp_{2n}(q)$ the result can be easily deduced from that for $H := SL_2(q^n)$. This is a special case of Proposition 3.3.

It is well known that there is an embedding of $e : H \rightarrow G$. Let T be the subgroup of H of order $q^n - 1$ (one can describe T as the matrix group $\{\text{diag}(a, a^{-1}) : a \in \mathbb{F}_{q^n}^\times\} < SL_2(q^n)$.

PROPOSITION 3.4. *Let $G = Sp_{2n}(q)$, $n > 1$, $(n, q) \neq (2, 2), (2, 3)$. Let $C = e(T)$. Let M be a non-trivial irreducible $\mathbb{F}G$ -module with central character ζ . Then C is self-centralizing and $M|_C$ contains the induced module $\text{Ind}_{Z(G)}^C(\zeta)$.*

Proof. The result follows from Proposition 3.3 for $SL_2(q^n)$ as long as we show that $e(T)$ is self-centralizing in G . It is well known that $e(T)$ is a maximal torus for G , which implies the claim, provided the torus is self-centralizing. However, we provide an elementary direct argument for this.

Note that the embedding e is obtained via a vector space embedding $\mathbb{F}_{q^n}^2 \rightarrow \mathbb{F}_q^{2n}$, induced from the embedding $\mathbb{F}_{q^n} \rightarrow \mathbb{F}_q^n$ when \mathbb{F}_{q^n} is viewed as a vector space over \mathbb{F}_q . It follows that $e(T)$ is reducible, and in fact a sum of two irreducible n -dimensional $\mathbb{F}_q e(T)$ -submodules W_1, W_2 , say. They are totally isotropic as $Sp_{2n}(q)$ contains no irreducible element of order $q^n - 1$. In this situation it is a standard fact that V has a basis $\{b_1, \dots, b_{2n}\}$ such that $b_i \in W_1, b_{n+i} \in W_2$ for $i = 1, \dots, n$, and $(b_i, b_{n+j}) = \delta_{ij}$. Then the representations of T on W_1, W_2 are dual to each other. As T is cyclic, the dual representation is obtained via the automorphism $t \rightarrow t^{-1}$ ($t \in T$). They are non-equivalent unless $n = 2, q = 2$, see Lemma 2.4. It follows from Schur's lemma that $C_G(e(T)) = e(T)$, as claimed. \square

Note that $Sp_4(2)$ is not simple, and $PSp_4(3) \cong SU_4(2)$. Proposition 3.4 is not true for $Sp_4(3)$, but Theorem 2.2 is true due to this isomorphism and Proposition 3.8 below. Note that $Sp_4(3)$ (but not $PSp_4(3)$) is an exception also for Proposition 1.8.

3.3. Groups G with $G/Z(G) = P\Omega_{2n}^+(q)$, $n > 3$, or $PSU_{2n}(q)$, $n > 1$

Set $r = q$ if $G/Z(G) = P\Omega_{2n}^+(q)$, and $r = q^2$ if $G/Z(G) = PSU_{2n}(q)$. Let V be a unitary space of dimension $2n > 2$ over \mathbb{F}_r , or an orthogonal of dimension $2n > 6$ and Witt index n over \mathbb{F}_r . This means that V has a totally isotropic subspace W of dimension n (totally singular, if V is orthogonal and r is even).

Let $B = \{b_1, \dots, b_{2n}\}$ be a basis of V such that $b_1, \dots, b_n \in W$ and the Gram matrix Γ corresponding to B is of shape

$$\Gamma = \begin{pmatrix} 0 & \text{Id}_n \\ \text{Id}_n & 0 \end{pmatrix},$$

Then $V = W \oplus W_1$, where $W_1 = \langle b_{n+1}, \dots, b_{2n} \rangle$ is a totally isotropic (singular) subspace of V . If B is fixed, $G(V)$ can be described as $\{g \in GL(V) : g\Gamma\sigma(g)^t = \Gamma\}$, except when V is orthogonal and $p = 2$.

Let P be the stabilizer of W in $G(V)$, and $H = \{g \in P : gW_1 = W_1\}$. Under the basis B , the matrices of H are of shape $\text{diag}(h, \sigma(h^t)^{-1})$, where $h \in GL_n(r)$. If V is orthogonal and q is odd then H is isomorphic to a subgroup of order 2 in $GL_n(r)$, see [18, Lemma 2.7.2]. If V is unitary then the determinant condition $\det h\sigma(h^t)^{-1} = 1$ implies $\sigma(\det(h)) = \det(h)$, whence $\det(h) \in U_1(q)$. Therefore, H is isomorphic to a normal subgroup of $GL_n(r)$ of index $q + 1$. In other cases $H \cong GL_n(r)$.

Set $X = \text{diag}(x, \sigma(x^t)^{-1})$, where $x \in GL(W)$ is irreducible of maximal order subject to the condition that $X \in G$. If V is orthogonal with q even then $|x| = r^n - 1$. In the unitary case $\det(x) \in \mathbb{F}_q$ implies $|x| = (r^n - 1)/(q + 1)$. Set $T = \langle X \rangle$.

LEMMA 3.5. *The subgroup T is self-centralizing.*

Proof. Let $\rho_1, \rho_2 : T \rightarrow GL_n(q)$ be the representations defined by $\rho_1(X) = x$ and $\rho_2(X) = \sigma(x^t)^{-1}$. Then ρ_1, ρ_2 are non-equivalent. Indeed, $\sigma(x^t)^{-1} = g x g^{-1}$ for some $g \in GL_n(r)$ if and only if $\sigma(x)^{-1} = h x h^{-1}$ for some $h \in GL_n(r)$ (as x and x^t are similar matrices). Hence the claim follows from Lemma 2.4.

By Schur's lemma, $C_{G(V)}(X) = \text{diag}(C, \sigma(C^{-1}))$, where C runs over $C_{GL_n(r)}(x)$, which is isomorphic to $F_{r^n}^\times$. So $C_{G(V)}(X)$ is cyclic. \square

With respect to the above basis matrices A are of shape:

$$\begin{pmatrix} \text{Id}_n & b \\ 0 & \text{Id}_n \end{pmatrix}, \quad (3.1)$$

where $b \in GL_n(r)$. We observe that H normalizes A and $C_H(A) = Z(G(V))$.

Obviously, A is an abelian group of exponent p . Let L, L_0 be as in (1) and (2) in Section 2 above.

LEMMA 3.6. *Let $a \in A$ be a matrix of shape (3.1).*

- (i) *Suppose that V is not an orthogonal space in characteristic 2. Then $a \in G(V)$ if and only if $b \in L$.*
- (ii) *Suppose that V is an orthogonal space in characteristic 2. Then $a \in G(V)$ if and only if $b \in L_0$.*
- (iii) *The mapping $M \rightarrow b$ is a bijection $A \rightarrow L$, unless $p = 2$ and $\sigma = 1$ when this is a bijection between A and L_0 .*
- (iv) *If V is orthogonal then $A < \Omega_{2n}^+(q)$.*

Proof. For (i)–(iii) see [7, p.240]. For (iv), suppose first that $p \neq 2$. Set $\Omega = \Omega_{2n}(q)$. As Ω is a normal subgroup in $G(V)$ of index 4, the result follows. Suppose that $p = 2$. In this case $|G(V) : \Omega| = 2$, so either $HA < \Omega$ or HA has a subgroup of index 2. As $p = 2$ and $n > 2$, H has no subgroup of index 2. So $A < \Omega$ unless A has an H -stable subgroup of index 2. This contradicts [9, Lemma 4.6], where it is shown that L_0 is an irreducible $\mathbb{F}_r H$ -module (provided that $n > 2$). \square

Note that $A < SU_n(q)$ in the unitary case.

LEMMA 3.7. *Every $P/Z(G)$ -orbit on $A \setminus \{1\}$ contains a regular $T/Z(G)$ -orbit.*

Proof. This follows from Lemma 2.8. Indeed, by Lemma 3.6(i), resp., 3.6(ii), A and L^+ , resp., L_0^+ , are isomorphic H -sets. (Here L^+ , resp. L_0^+ , is the additive group of L , resp. L_0 .) The action of H on A by conjugation corresponds to the congruence action of $GL_n(r)$ on L , resp. L_0 , and hence we may use Lemma 2.8. \square

Recall that $D_\sigma = \{g \in Z(GL_n(r)) : g\sigma(g) = 1\}$, see Lemma 2.5. Let D be the image of D_σ in H under the isomorphism $GL_n(r) \rightarrow H$. It is clear that $D \leq Z(G(V))$.

PROPOSITION 3.8. *Let $G/Z(G) = P\Omega_{2n}^+(q)$ or $PSU_{2n}(q)$. Let M be a non-trivial irreducible $\mathbb{F}G$ -module with central character ζ . Then $M|_T$ contains a submodule isomorphic to $\text{Ind}_{Z(G)}^T(\zeta)$. In particular, if $\zeta = 1_{Z(G)}$ then $M|_T$ contains a regular submodule, and hence $1_T \in M|_T$.*

Proof. This follows from Lemmas 2.3 and 3.7. Indeed, as $(p, |Z(G)|) = 1$, there is a p -subgroup A_1 in G which projects to A under the homomorphism $G \rightarrow \Omega_{2n}(q)$, and the preimage of A coincides with $A_1 Z(G)$. If T_1 , resp., S , is the preimage of T , resp. $SL_n(q)$, in G then A_1 is T_1 - and S -invariant. Moreover, $Z(G)$ is in the kernel of the conjugation action of T_1 and S on A_1 , so A_1 can be viewed as $T_1/Z(G)$ and $S/Z(G)$ -sets. Then A and A_1 are isomorphic with respect to these actions.

We can decompose M as a direct sum of homogeneous A_1 -modules $M = \bigoplus_{\alpha \in \text{Irr } A_1} M_\alpha$. Obviously, this decomposition can be arranged as follows: $M = \bigoplus_O (\bigoplus_{\alpha \in O} M_\alpha)$, where the O 's are $N_G(A_1)$ -orbits on $\text{Irr}(A_1)$. As $[S, A_1] \neq 1$, it follows that there exists $\alpha \in \text{Irr}(A_1)$, $\alpha \neq 1_{A_1}$ such that $S\alpha \neq \alpha$. Note that the conjugation action of S on A_1 is realized via the congruence action of $SL_n(r)$ on L or L_0 (see Section 2). Fix this α , and let $O_1 = S\alpha$ be the S -orbit of α . By Lemma 2.8, there is $\beta \in O_1$ such that $t\beta = \beta$ for $t \in T$ implies $t \in D$. As $D \leq Z(G(V))$, it follows that $t\beta = \beta$ for $t \in T_1$ implies $t \in Z(G)$. Thus, $C_{T_1}(\beta) = Z(G)$. Let $\beta|_{Z(G)} = \zeta$. Then, by Lemma 2.3, $M|_T$ contains a submodule isomorphic to $\text{Ind}_{Z(G)}^T(\zeta)$. \square

3.4. The groups G with $G/Z(G) = \Omega_{2n+1}(q)$, $n > 2$, q odd, or $PSU_{2n+1}(q)$, $n \geq 1$

Let V be a unitary or orthogonal space of dimension $2n+1$ over \mathbb{F}_r , where $r = q^2$ in the unitary case, and $r = q$ otherwise. Let W be a maximal totally isotropic subspace of V , so $\dim W = n$. Let $G_1 = \Omega_{2n+1}(q)$ in the orthogonal case, and $G_1 = SU_{2n+1}(q)$ in the unitary case.

We wish to mimic the reasoning in Subsection 3.3; in particular, the groups A, T here are analogous to those above. However, Lemma 3.7 is not true for our current situation, as $C_T(A)$ is not necessarily in $Z(G)$.

Let $B = \{b_1, \dots, b_{2n+1}\}$ be a basis of V such that $b_1, \dots, b_n \in W$ and the Gram matrix Γ corresponding to B is of shape

$$\begin{pmatrix} 0 & 0 & \text{Id}_n \\ 0 & 1 & 0 \\ \text{Id}_n & 0 & 0 \end{pmatrix}$$

Then $V = W \oplus V_0 \oplus W_1$, where $V_0 = \langle b_{n+1} \rangle$ and $W_1 = \langle b_{n+2}, \dots, b_{2n+1} \rangle$. So $V_0^\perp = W + W_1$. By definition, $G(V) = \{g \in GL_{2n+1}(r) : g\Gamma\sigma(g)^t = \Gamma\}$. Let H be the stabilizer in G_1 of each of the subspaces W, W_1, V_0 of this decomposition. Then H consists of matrices

$$\begin{pmatrix} h & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & \sigma(h^t)^{-1} \end{pmatrix}, \quad (3.2)$$

where $h \in GL_n(r)$ and $f \in G(V_0)$. If V is orthogonal then $H < SO_{2n+1}(q)$, whence $f = 1$. Note that V_0^\perp has Witt index n . Therefore, $O(V_0^\perp) = O_{2n}^+(r)$, and hence H is isomorphic to a subgroup of index 2 in $GL_n(r)$. If V is unitary then $f \cdot \det h \cdot \det \sigma(h^{-1}) = 1$. It follows that H is isomorphic to $GL_n(r)$.

Let $x \in GL(W)$ be a Singer cycle, so $|x| = r^n - 1$. We set

$$X = \begin{pmatrix} x & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & \sigma(x^t)^{-1} \end{pmatrix} \quad \text{and} \quad T = \langle X \rangle,$$

where $e = \sigma(\det x) \cdot \det x^{-1}$. So T is a cyclic subgroup of $G(V)$. Note that $x^{(r^n-1)/(r-1)}$ generates the center of $GL_n(r)$ and $C_{GL_n(r)}(x) = \langle x \rangle$.

LEMMA 3.9. *Let $T = \langle X \rangle$. Then one of the following holds:*

- (i) $C_{G_1}(X) = T$.
- (ii) V is unitary of dimension 3 and $q = 2$ or 3.

Proof.

Suppose that (ii) does not hold. It follows from Lemma 2.4 that the representations $h \rightarrow h$ and $h \rightarrow \sigma(h^t)^{-1}$ are not equivalent. By Schur's lemma, $C_{G(V)}(X) = \text{diag}(A, b, C)$, where $C = \sigma(A^t)^{-1}$, A runs over $C_{GL_n(r)}(x) = \langle x \rangle$ and $b \in G(V_0)$. Then $C_{G(V)}(X)$ is obviously abelian. If V is orthogonal then $\det A \cdot \det C = 1$, and hence $b = 1$. So (i) follows. Suppose that V is unitary. Then $b \cdot \det A \cdot \sigma(\det C) = 1$. As $A = x^k$ for some k by Schur's lemma, we have $C = \sigma(x^t)^{-k}$, and hence $b = \det x^{-k} \sigma(x)^k = e^k$, as required. \square

With respect to the above basis consider the subgroups A, U, D of G , consisting of matrices, respectively, of shape

$$\begin{pmatrix} \text{Id}_n & 0 & \alpha \\ 0 & 1 & 0 \\ 0 & 0 & \text{Id}_n \end{pmatrix}, \quad \begin{pmatrix} \text{Id}_n & \beta & \gamma \\ 0 & 1 & -\sigma(\beta)^t \\ 0 & 0 & \text{Id}_n \end{pmatrix}, \quad \begin{pmatrix} \lambda \cdot \text{Id}_n & 0 & 0 \\ 0 & \lambda^{-2n} & 0 \\ 0 & 0 & \lambda \cdot \text{Id}_n \end{pmatrix}, \quad (3.3)$$

where $\lambda \in F_r$ and $\sigma(\lambda)\lambda = 1$. In particular, if V is an orthogonal space then $\lambda^2 = 1$, and hence $|D| \leq 2$ and $D < T$. (It follows from [18, 2.7.2 and 2.5.13] that $|D| = 1$ if and only if $(q-1)n/2$ is even.)

Suppose that V is unitary. Then $\lambda \in U_1(q)$. We write $a = a(\alpha)$ for $a \in A$ and $u = u(\beta, \gamma)$ for $u \in U$. Simple computation shows that $u = u(\beta, \gamma) \in U$ is equivalent to $\gamma + \sigma(\gamma^t) + \beta^t \sigma(\beta) = 0$. In particular, $a(\alpha) \in A$ is equivalent to $\alpha + \sigma(\alpha^t) = 0$. Note that $D = \langle Y \rangle$, where $Y = X^{(r^n-1)/(q+1)}$. The matrices in D are of shape $\lambda \text{diag}(\text{Id}, \lambda^{-2n-1}, \text{Id})$, so $|D| = q+1$. In addition, D contains $Z(G_1)$.

It is obvious that H normalizes A . In addition, $A = Z(U) = U'$, where U' is the derived group of U (see [7, Lemma 3.1]).

LEMMA 3.10. *$D = C_H(A)$ and $D = C_T(A)$. In addition, $C_D(u) = Z(G_1)$ for every $u \in (U \setminus A)$.*

Proof. The first statement means that D is the kernel of the conjugation action of H on A . As explained in Section 2, H acts on A via the action of $GL_n(r)$ on L , so the claim follows from Lemma 2.5. The second statement follows as $D < T$. The additional assertion is clear from the shape of matrices of D and U . \square

Define H_0 to be the subgroup of H whose matrices are $\text{diag}(h, 1, \sigma(h^t)^{-1})$ with $h \in SL_n(r)$.

LEMMA 3.11. (i) Suppose that $u \in (U \setminus A)$ and $g \in D$ is non-scalar. Then $[g, u] \notin A$. (This means that the D -orbits on $U \setminus A$ are isomorphic to each other and have size $|D/Z(G_1)|$.)

(ii) If $1 \neq a \in A$, then there exists $h \in H_0$ such that $C_T(hah^{-1}) = D$.

(iii) Let χ be an irreducible character of A , $\chi \neq 1_A$. Then there exists $g \in H_0$ such that $gTg^{-1} \cap C_H(\chi) = D$.

Proof. (i) is obvious from the shape of the matrices above.

(ii). Suppose first that $n = 1$. Then $C_H(a)$ consists of matrix of shape satisfying $h\alpha\sigma(h) = \alpha$, or $h\sigma(h) = 1$, that is, $h \in U_1(q)$. It follows that $C_H(a) = D$, that is, (ii) holds for $g = 1$.

Suppose $n > 1$. Then $C_H(a)$ consists of matrix of shape (3.2) satisfying $h\alpha\sigma(h^t) = \alpha$, or $h\alpha = \alpha\sigma(h^t)^{-1}$. As $h \rightarrow \sigma(h^t)^{-1}$ is an irreducible representation of $GL_n(q)$, by Schur's lemma α is invertible. As $\alpha + \sigma(\alpha) = 0$, the matrix α can be regarded as a matrix of a non-degenerate unitary form on W , and the condition $h\alpha\sigma(h^t) = \alpha$ determines a unitary group preserving the form. Now $U_n(q)$ is a proper subgroup of $GL_n(q^2)$, and $SU_n(q) \neq SL_n(q^2)$.

This conclusion is necessary in order to use Lemma 2.7. Consider the H_0 -orbit $\Omega = \{hah^{-1} : h \in H_0\}$, and $Z_1 = \ker \Omega := \{h \in H_0 : h\omega = \omega \text{ for all } \omega \in \Omega\}$.

Recall that every normal subgroup of $GL_n(r)$ that does not contain $SL_n(r)$ belongs to the center of $GL_n(r)$, unless $(n, r) = (2, 2), (2, 3)$. As here r is a square in the unitary case and $n > 2$ in the orthogonal case, these exceptions do not occur. So $Z_1 \leq Z(H_0)$, and hence Z_1 consists of matrices of the shape $\text{diag}(s \cdot \text{Id}, s^{-1}\sigma(s), \sigma(s)^{-1} \cdot \text{Id})$, where $0 \neq s \in \mathbb{F}_r$. As in the case $n = 1$ above, one observes that $Z_1 = D$. Thus, H_0/D acts faithfully on Ω . Let Ω_1 denote the set of Z_1 -orbits on Ω . As Ω is a transitive H_0 -set, it follows that all $Z(H_0)$ -orbits are isomorphic to each other, and have $Z_1 = D$ in the kernel, whence each is the regular $Z(H_0)/D$ -set. In addition, H_0 permutes these transitively. Applying Lemma 2.7 to Ω_1 , we find some $\omega_1 \in \Omega_1$ such that the orbit $(T/Z(H_0)) \cdot \omega_1$ is regular. This means that $t\omega_1 = \omega_1$ for $t \in T$ implies $t \in Z(H_0)$. Next pick any $\omega \in \omega_1$. As $Z(H_0)/Z_1$ acts regularly on ω_1 , it follows that $t\omega = \omega$ implies $t \in Z_1$. As Z_1 is in the kernel of Ω , the orbit $T\omega$ is a regular T/Z_1 -orbit. As $\omega = gag^{-1}$ for some $g \in H_0$, the result follows.

(iii) follows from (ii) as $\text{Irr}(A)$ and A are isomorphic permutation sets for $\text{Aut } A$. \square

REMARK 3.12. If V is orthogonal and $n(q-1)/2$ is even then $D = 1$ as mentioned above. If V is unitary then $D = Z(G)$ if and only if $|Z(G)| = q+1$, and if and only if $q+1$ divides $2n+1$. Therefore, in this case Proposition 1.8 follows from Lemma 3.11.

PROPOSITION 3.13. Let $G/Z(G) = P\Omega_{2n+1}(q)$. Let M be a non-trivial irreducible $\mathbb{F}G$ -module with central character ζ . Then $M|_T$ contains a submodule isomorphic to $\text{Ind}_{Z(G)}^T \zeta$. In particular, if $\zeta = 1_{Z(G)}$ then $M|_T$ contains a regular submodule, and hence $1_T \in M|_T$.

Proof. By the above, T acts faithfully on $V_0^\perp = W + W_1$. Let $0 \neq v \in V_0$, and let $K = C_G(v)$. Then $K \cong \Omega(V_0^\perp)$. As the Witt index of V_0^\perp equals n , it follows that $O(V_0^\perp) \cong O_{2n}^+(q)$. Obviously, $[D, K] = 1$. If $D = 1$ then the result follows from Lemma 3.8.

Let $|D| = 2$. Then $M = M_1 \oplus M_2$, where D acts trivially on M_1 . As D is not scalar in G , both M_1, M_2 are non-zero and K -stable.

We show that $K|_{M_i}$ is non-trivial for $i = 1, 2$. Let M' denote the subspace spanned by all $(a - \text{Id})M$ for $a \in A$. This space is stable under $C_G(A)$, in particular, under $U = O_p(P)$ and D . It suffices to show that D is not scalar in M' . Suppose the contrary. Let $\tau(g) = g|_{M'}$ for $g \in C_G(A)$. Obviously, $\ker \tau \cap A = 1$. It follows that $\ker \tau \cap U = 1$ (otherwise, $[U, \ker \tau] \leq A \cap \ker \tau = 1$, and hence $\ker \tau \leq Z(U)$, contrary to Lemma 3.11(1)). Then $\tau([d, u]) \neq 1$ for

$u \in (U \setminus A)$, $1 \neq d \in D$ by 3.11(3). Therefore, $\tau(d)$ is not scalar, and hence d has eigenvalues 1 and -1 on M' .

To complete the proof let $M_i = \sum_O \sum_{\alpha \in \text{Irr } A} M_i^\alpha$, where $M_i^\alpha = \{m \in M_i : am = \alpha(a)m \text{ for all } a \in A\}$. Then use Lemma 3.8. \square

LEMMA 3.14. *Let $X = YE$, where E is an extraspecial p -group normal in X , and $Y = \langle y \rangle$ is a cyclic group and let $Z = C_Y(E)$. Suppose that $[Y, Z(E)] = 1$ and $C_Y(e) = Z$ for every $e \in E \setminus Z(E)$. Let M be a faithful irreducible $\mathbb{F}X$ -module non-trivial on $[E, E]$. Let $M|_Z = \lambda \cdot \text{Id}$. Then either $M = M_1 + N$ or $M_1 = M + N$, where $\dim N = 1$, $M_1 = m \cdot \text{Ind}_Z^Y \lambda$, and $m + \delta = \dim M/|Y/Z|$ for some $\delta \in \{1, -1\}$.*

Proof. This is a small refinement of [8, Theorem 9.18]. Indeed, the lemma coincides with [8, Theorem 9.18] if $Z = 1$. In general, let $k = |Y/Z|$ and let s be a scalar matrix of order $|Y|$ such that $s^k = y^k$. Set $S := \langle s^{-1}y \rangle$ and $X_1 = \langle E, S \rangle$. Then X_1 is a semidirect product of E and S , and $C_S(E) = 1$. In addition, all non-trivial orbits of S on $E/Z(E)$ are of the same size as S . This means that SE satisfies the assumptions of [8, Theorem 9.18], saying that in this case M_1 is a free $\mathbb{F}S$ -module of rank m with m as above. As the element y is a scalar multiple of $s^{-1}y$, our conclusion on $M|_Y$ follows from the result about $M|_S$. \square

LEMMA 3.15. *Let τ be an irreducible representation of the group DU nontrivial on U' . Then every irreducible representation λ of D such that $\lambda|_{Z(G_1)} = \zeta \cdot \text{Id}$ is a constituent of $\tau|_D$, unless $Z(G_1) = 1$ and $\dim \tau = q$.*

Proof. Recall that $C_D(U) = Z(G_1)$, see Lemma 3.10. Let $U_1 = \{u \in U : \tau(u) \text{ is scalar}\}$. Then $U \neq U_1$. Set $E = \tau(U)$, and $x = |E/Z(E)|$. By [7, Lemma 3.13], $E = Z(E) \cdot E_1$, where E_1 is an extraspecial group and x is a q -power. As τ is irreducible, $\tau(U_1) = Z(E)$. Let $x = r^k = q^{2k}$. Obviously, $|E_1/Z(E_1)| = x$. It is well known that an irreducible representation of E_1 is either one-dimensional or of degree \sqrt{x} , in our case this is q^k . So $\dim \tau = q^k$ for some k (this is also stated in [12, Corollary 12.6]). In addition, E_1 can be chosen D -stable. Indeed, U/A is an $F_p D$ -module and U_1/A is obviously a submodule. By Maschke's theorem, there is a D -stable complement U_2/A . Then $E_1 = \tau(U_2)$ is D -stable.

Note that $U_2/A \cong E_1/Z(E_1)$, and this is an $F_p D$ -module isomorphism. By Lemma 3.11(1), the D -orbits on U/A are of size $|D/Z(G_1)|$, and hence so are the D -orbits on $E_1/Z(E_1)$. As D acts trivially on A and $\tau(A)$, we may apply Lemma 3.14 in order to claim that $\tau|_D$ contains a submodule isomorphic to $\text{Ind}_{Z(G_1)}^D(\zeta)$, unless $\dim \tau + 1 = |D/Z(G_1)|$. As $|D| = q + 1$ (see comments prior to Lemma 3.10), the lemma follows. \square

PROPOSITION 3.16. *Let $G = SU_{2n+1}(q)$, where $(2n+1, q+1) \neq 1$. Let $T < G$ be as in Lemma 3.9. Let ϕ be a non-trivial irreducible \mathbb{F} -representation of G with central character ζ . Then the restriction $\phi|_T$ contains all irreducible representations of $\tau \in \text{Irr}(T)$ such that $\tau|_{Z(G)} = \zeta \cdot \text{Id}$.*

Proof. Now $G = G_1$. Let H, A be as above. Recall that H acts on A by conjugation, and this action translates in the usual way to an action of H on $\text{Irr}(A)$ by setting $\alpha^n(a) = \alpha(nan^{-1})$ for $n \in H$, $a \in A$ and $\alpha \in \text{Irr}(A)$. Let M be an $\mathbb{F}G$ -module afforded by ϕ . For $\alpha_i \in \text{Irr}(A)$ set $M_\alpha = \{m \in M : gm = \alpha(g)m \text{ for all } g \in A\}$. We can write

$$M|_A = \bigoplus_O \bigoplus_{\alpha \in O} M_\alpha,$$

where O runs over the orbits of H in $\text{Irr}(A)$. For any orbit O , the subspace $M_O = \sum_{\alpha \in O} M_\alpha$ is an H -submodule of $M|_H$. Since $A \cap Z(G) = 1$, $M|_A$ is a faithful A -module. Therefore, there exists an orbit O such that A acts non-trivially in M_O , and we fix this O from now on. Then A acts non-trivially on every M_α for $\alpha \in O$. It is easy to observe that $A = U' = Z(U)$, see, for instance, [7, Lemma 3.1]. In addition, $[D, A] = 1$ (Lemma 3.10). Therefore, every M_α is DU -stable for $\alpha \in O$.

By Lemma 3.11(2), the group T/D has a regular orbit in O . Let $\alpha_0 \in O$ be such that the T -orbit O' is of size $|T/D|$. Set $M' := \bigoplus_{\alpha \in O'} M_\alpha$.

Observe that D acts trivially on $\text{Irr}(A)$. Therefore, the action of T on M_O is realized via the action of D inside each M_α and the action of T/D on the M_α 's for $\alpha \in O'$, which are regularly permuted by T/D . Let N be an irreducible submodule of $M'|_{DU}$. Then $A = U'$ is non-trivial on N . As shown in the proof of Lemma 3.15, $\text{Ind}_{Z(G)}^D(\zeta)$ is a submodule of $N|_D$. Then $\text{Ind}_{Z(G)}^D(\zeta)$ is a submodule of $M'|_T$, and the result follows. (We also use ζ to denote the module afforded by the character ζ .) \square

Proof of Proposition 1.8. For the groups $SL(n, q)$ and $P\Omega_{2n}^-(q)$, $n > 3$, the result is contained in Lemma 3.2, whereas for $PSU_{2n}(q)$, $n > 1$, and $P\Omega_{2n}^+(q)$, $n > 3$, this follows from Lemmas 3.5 and 3.7. See Remark 3.12 for $PSO_{2n+1}(q)$ with $n(q-1)/2$ odd.

3.5. The exceptional case

In this subsection we assume that $G = SU_n(q)$, where $n \geq 3$ is odd and coprime to $q+1$. This implies that $Z(G) = 1$. As above, we set $r = q^2$. Let P be the stabilizer of an isotropic line W at the natural G -module V . Choose any complement V_1 to W in W^\perp . Then there exists a basis b_1, \dots, b_n of V whose Gram matrix is

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & \text{Id}_{n-2} & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then $b_1 \in W$ and $b_2, \dots, b_{n-1} \in W^\perp$. Set $V_1 = \langle b_2, \dots, b_{n-1} \rangle$. Let H, U be the subgroups of G consisting of matrices of shape

$$\begin{pmatrix} \sigma(a^{-1}) & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & a \end{pmatrix}, \quad \begin{pmatrix} 1 & -\sigma(v^t) & w \\ 0 & \text{Id}_{n-2} & v \\ 0 & 0 & 1 \end{pmatrix},$$

where $0 \neq a \in \mathbb{F}_r$, $y \in U(V_1)$ and $v \in V_1$. Here v is an arbitrary element of \mathbb{F}_r^{n-2} . The entry w depends on v but we do not need to express the dependence explicitly; the determinant condition is $a\sigma(a^{-1}) \det y = 1$.

Note that $Z(U)$ consists of all matrices in U with $v = 0$. The conjugation action of H on U induces on $U/Z(U)$ the structure of $\mathbb{F}_r H$ -module. Moreover, the subgroup $\{\text{diag}(1, h, 1) : h \in SU(n-2, q)\}$ acts on $U/Z(U)$ exactly as on V_1 .

It is well known that $U(V_1) \cong U_{n-2}(q)$ contains a self-centralizing cyclic subgroup $\langle t \rangle$ of order $q^{n-2} + 1$. Set $T = \langle X \rangle$, where $X = \text{diag}(a, t, \sigma(a^{-1}))$ and a is a generator of \mathbb{F}_r^\times such that $\det X = 1$. As $\sigma(a) = a^q$, we have $\det t = a^{q-1}$.

LEMMA 3.17. (i) $T = C_G(X) = C_H(X)$ is a cyclic p' -group.

(ii) If $1 \neq c \in T$, $u \in (U \setminus Z(U))$ then $[c, u] \notin Z(U)$. In other words, every non-identity element $c \in C$ acts on $U/Z(U)$ fixed point freely.

Proof. The matrix X gives rise to three irreducible representations of T , namely, $X \rightarrow t$, $X \rightarrow a$ and $X \rightarrow a^{-q}$. As $a^{q+1} \neq 1$, these are pairwise non-equivalent. (If $n = 3$ then $t \in \mathbb{F}_r$ and $t^{q+1} = 1$; so the claim is true for $n = 3$ as well.) Then, by Schur's lemma, the elements of $C_{G(V)}(X)$ are of shape $\text{diag}(b^{-q}, t^i, b)$ for some i and $b \in \mathbb{F}_r^\times$. So $(|T|, p) = 1$.

(ii) Suppose the contrary. Then there is $0 \neq v \in V_1$ such that $t^i b^{-1} v = v$. Hence v is a b -eigenvector for t^i . Therefore t stabilizes the b -eigenspace of t^i on V_1 . As t is irreducible on V_1 , it follows that the b -eigenspace coincides with V_1 , and hence $t^i = b \cdot \text{Id}$. Then $t^i \in Z(U(V_1))$, whence $b^{q+1} = 1$ and $b^{-q} = b$. This means that c is scalar, a contradiction.

(i) If C is not cyclic then C contains a non-cyclic elementary abelian s -subgroup of order s^2 for some prime s . It is easy to observe that this contradicts (ii). \square

PROPOSITION 3.18. *Let $G = SU_n(q)$, where $n \geq 3$ is odd and coprime to $q + 1$, and let T be as above. Let ϕ be an irreducible representation of G . Then $\phi|_T$ contains every irreducible representation of T as a constituent, unless, possibly, ϕ is a Weil representation of G .*

Proof. Let P be the stabilizer of a one-dimensional isotropic subspace at the natural G -module. Then P is a parabolic subgroup of G . Let U be the unipotent radical of P . By [12, Corollary 12.4], the character $\phi|_U$ contains a non-trivial linear character χ , say, of U , unless $\phi = 1_G$ or a Weil representation of G .

Now $\chi|_{Z(U)}$ is trivial, since $Z(U) = U'$, and hence χ can be viewed as a character of $U/Z(U)$. Denote the character group of $U/Z(U)$ by Ω , so $\chi \in \Omega$. The action of T on Ω is dual to that on $U/Z(U)$. By Maschke's theorem, $U/Z(U)$ is a completely reducible $\mathbb{F}_r T$ -module. It follows from Lemma 3.17(2) that every element of T acts fixed point freely on $\Omega \setminus \{1\}$, in particular, the T -orbit of χ is of length $|T|$.

Let M be the module afforded by the representation ϕ , and let M' be the subset of fixed vectors for $Z(U) = U'$. Then χ is the character of a constituent of the restriction $M'|_U$, and obviously, $TM' = M'$. Therefore, $M' = \bigoplus_{\omega \in \Omega} M_\omega$, where $M_\omega := \{m \in M' : um' = \omega(u)m'\}$. As $M_\chi \neq 0$, we get a non-trivial T -module $N := \bigoplus_{c \in T} M_{c\chi}$, where $\chi \rightarrow c\chi$ ($c \in T$) is the action of T on Ω defined above. Moreover, $N|_T$ is regular as the T -orbit of χ is regular and $Z(G) = 1$ (see Lemma 2.3). It follows that $N|_T$ contains every irreducible T -module. \square

Next we establish the converse of Proposition 3.18 in the case of complex representations.

Recall that the generic Weil character $\omega_{n,q}$ of $U_n(q)$ is defined by

$$\omega_{n,q} : h \mapsto (-1)^n (-q)^{\dim_{\mathbb{F}_q} \text{Ker}(h-1)},$$

where the dimension in the exponent is on the natural module for G over \mathbb{F}_{q^2} . For $n > 2$ this is the sum $\sum_{l=0}^q \omega_{n,q}^l$ of $q + 1$ irreducible characters $\omega_{n,q}^l$, $0 \leq l \leq q$ (we refer to them as irreducible Weil characters).

THEOREM 3.19. *Let $G = SU_n(q)$, where $n \geq 3$ is coprime to $2(q + 1)$, and let T be the maximal torus of G of order $(q - 1)(q^{n-2} + 1)$ described above.*

(i) *If χ is any nontrivial non-Weil irreducible character of G , then $\chi|_T$ contains every irreducible character of T .*

(ii) *If χ is any of the $q + 1$ Weil irreducible characters of G , then $\chi|_T$ contains all but $(q - 1)$ irreducible characters of T .*

(iii) *If $\mu \in \text{Irr}(T)$, then $\text{Ind}_T^G(\mu)$ contains all but possibly one nontrivial irreducible character of G , and the missing character must be a Weil character of G . The total number of $\mu \in \text{Irr}(T)$ such that $\text{Ind}_T^G(\mu)$ misses a nontrivial irreducible character of G is $q^2 - 1$.*

(iv) The only irreducible character of G which is not a constituent of $\text{Ind}_T^G(1_T)$ is the (unipotent) Weil character of G , of degree $(q^n - q)/(q + 1)$.

Proof. 1) Note that (i) is already proved in Proposition 3.18. For brevity, set $q_1 := q - 1$ and $q_2 := (q^{n-2} + 1)/(q + 1)$. Recall that $T = G \cap T_1$, where T_1 is a maximal torus of $U_n(q)$. Let V be the natural module for $U_n(q)$ and $\bar{V} := V \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$. One observes that T_1 can be diagonalized under a suitable basis of \bar{V} as follows:

$$T_1 = \left\{ g_{i,j} := \text{diag}(a^i, a^{-q_1}, b^j, b^{-q_2}, \dots, b^{q^{n-3}j}) \mid 0 \leq i \leq q^2 - 2, 0 \leq j \leq q^{n-2} \right\}.$$

Here, $a, b \in \bar{\mathbb{F}}_q^\times$ are some fixed elements of order $q^2 - 1$, respectively $q^{n-2} + 1$, chosen such that $a^{q_1} = b^{q_2} =: c$. We also fix $\alpha, \beta \in \mathbb{C}^\times$ of order $q^2 - 1$, respectively $q^{n-2} + 1$, such that $\alpha^{q_1} = \beta^{q_2} =: \gamma$. Then every irreducible character of T_1 is of the form $\lambda = \lambda_{s,t} : g_{i,j} \mapsto \alpha^{si} \beta^{tj}$ with $0 \leq s \leq q^2 - 2, 0 \leq t \leq q^{n-2}$.

2) The Weil irreducible characters are of shape

$$\omega_{n,q}^k : h \mapsto \frac{(-1)^n}{q+1} \sum_{l=0}^q \gamma^{kl} (-q)^{\dim_{\mathbb{F}_q} \text{Ker}(h - c^l)},$$

cf. [29]. (Here, $\text{Ker}(h - c^l)$ is computed on the natural module V for G .) Observe that

$$\dim_{\mathbb{F}_q} \text{Ker}(g_{i,j} - c^l) = \begin{cases} 0, & q_1 \nmid i, q_2 \nmid j, \\ 2\delta_{i_1,l}, & i = q_1 i_1, q_2 \nmid j, \\ (n-2)\delta_{j_2,l}, & q_1 \nmid i, j = q_2 j_2, \\ 2\delta_{i_1,l} + (n-2)\delta_{j_2,l}, & i = q_1 i_1, j = q_2 j_2, \end{cases}$$

where $i_1, j_2 \in \mathbb{Z}$ and $0 \leq i_1, j_2 \leq q$. It follows that

$$\begin{aligned} & [\omega_{n,q}^k|_{T_1}, \lambda_{s,t}]_{T_1} = \\ & -\frac{1}{(q+1)|T_1|} \sum_{l=0}^q \gamma^{kl} \cdot \left(\sum_{i=0}^{q^2-2} \alpha^{-si} + (q^2-1)\gamma^{-sl} \right) \cdot \left(\sum_{j=0}^{q^{n-2}} \alpha^{-tj} - (q^{n-2}+1)\gamma^{-tl} \right) = \\ & -\frac{1}{q+1} \sum_{l=0}^q \gamma^{kl} \cdot (\delta_{0,s} + \gamma^{-sl}) \cdot (\delta_{0,t} - \gamma^{-tl}) = \delta_{0,s} \delta_{0,k-\bar{t}} + \delta_{0,k-\bar{s}+\bar{t}} - \delta_{0,s} \delta_{0,t} \delta_{0,k} - \delta_{0,t} \delta_{0,k-\bar{s}}, \end{aligned}$$

where for each $i \in \mathbb{Z}$ we choose $0 \leq \bar{i} \leq q$ such that $q+1$ divides $(i - \bar{i})$. Thus $\lambda_{s,t}$ is a constituent of $\omega_{n,q}^k|_{T_1}$ precisely when $k = \bar{s} + \bar{t}$ and $t \neq 0$, in which case this multiplicity is 1 if $s \neq 0$ and 2 if $s = 0$.

3) So far we have used only the assumption that n is odd. Now we take into account the hypothesis $(n, q+1) = 1$, which implies that $T_1 = T \times Z$ for $Z := Z(U_n(q))$. Clearly,

$$T = \{g_{i,j} \in T_1 \mid \bar{i} = \bar{j}\},$$

and every $\mu \in \text{Irr } T$ can be obtained by restricting some $\lambda_{s,t}$ to T .

We claim that $(\lambda_{s,t})_T = (\lambda_{s',t'})_T$ precisely when there exist $x, y \in \mathbb{Z}$ such that

$$s' = s + q_1 x, \quad t' = t + q_2 y, \quad (q+1)|(x+y).$$

Indeed, assume that $\lambda_{s,t}|_T = \lambda_{s',t'}|_T$. Evaluating it at $g_{0,q+1} \in T$, we get $\beta^{(t'-t)(q+1)} = 1$, whence $t' = t + q_2 y$ for some $y \in \mathbb{Z}$. Similarly, by evaluating at $g_{q+1,0} \in T$ we get $s' = s + q_1 x$ for some $x \in \mathbb{Z}$. Finally, by evaluating at $g_{1,1} \in T$ we obtain $1 = \alpha^{s'-s} \beta^{t'-t} = \gamma^{x+y}$ and so $q+1$ divides $x+y$. It is easy to check that the converse of our claim holds.

Since $T_1 = T \times Z$, each $\mu \in \text{Irr } T$ has precisely $q + 1$ extensions $\lambda = \lambda_{s,t}$ to T_1 , which are uniquely determined by their restrictions to Z , i.e. by $\overline{s+t}$.

4) Recall that $G = SU_n(q)$ (with $n \geq 3$) has exactly $q + 1$ Weil irreducible characters, which can be obtained by restricting $\omega_{n,q}^k$, $0 \leq k \leq q$, to G .

Now suppose that $\mu \in \text{Irr } T$ does not enter $\omega_{n,q}^k|_T$ for some k , $0 \leq k \leq q$. By the previous observation, we can find an extension $\lambda_{s,t}$ of μ so that $k = \overline{s+t}$. By the assumption, $\lambda_{s,t}$ cannot enter $\omega_{n,q}^k|_T$. This implies by the conclusion of 2) that $t = 0$.

Conversely, suppose that $\mu = \lambda_{s,0}|_T$ for some s . We claim that μ is a constituent of $\omega_{n,q}^l|_T$ if and only if $l \neq \overline{s}$. Indeed, let $k := \overline{s}$. Now if μ enters $\omega_{n,q}^k|_T$, then by the conclusion of 2) we must have that $\mu = \lambda_{u,v}|_T$ for some $u, v \in \mathbb{Z}$ with $k = \overline{u+v}$ and $v \neq 0$. Thus, $\lambda_{s,0}$ and $\lambda_{u,v}$ are two extensions to T_1 of μ with $\overline{s} = k = \overline{u+v}$. By the last observation in 3), these two extensions are the same, whence $v = 0$, a contradiction. Next we consider any $l \neq k$, $0 \leq l \leq q$. Again by the last observation in 3) we can find an extension $\lambda_{s',t'}$ of μ to T_1 with $l = \overline{s'+t'}$. It follows by the discussions in 3) that $s' = s + q_1x$, $t' = q_2y$, and $(q+1)$ divides $(x+y)$. Notice that $q_1 \equiv -2 \pmod{q+1}$ and $q_2 \equiv n-2 \pmod{q+1}$. Hence

$$l - k \equiv (s' + t') - s = q_2y + q_1x \equiv ny \pmod{q+1}.$$

Since $l \not\equiv k \pmod{q+1}$ and $(n, q+1) = 1$, we must have that $(q+1) \nmid y$ and so $t' \neq 0$. Applying the results of 2), we see that $\lambda_{s',t'}$ is a constituent of $\omega_{n,q}^l|_{T_1}$, and so μ is a constituent of $\omega_{n,q}^l|_T$.

To complete the proof of (iii), observe that the $q^2 - 1$ characters $\lambda_{s,0}$ of T_1 have pairwise distinct restrictions to T . Indeed, suppose $\lambda_{s,0}|_T = \lambda_{s',0}|_T$. Then by 3), $s' - s = q_1x$ with $(q+1)$ divides x , whence $s' = s$.

To obtain (iv), just notice that $1_T = \lambda_{0,0}|_T$, and so the only irreducible character of G which is not contained in $\text{Ind}_T^G(1_T)$ is $\omega_{n,q}^0$.

5) To prove (ii), consider any Weil character $\chi = \omega_{n,q}^k|_G$ of G . We have shown in 4) that $\mu \in \text{Irr } T$ does not enter χ_T precisely when $\mu = \lambda_{s,0}|_T$ with $k = \overline{s}$. Since $0 \leq s \leq q^2 - 2$, there are exactly $q - 1$ possibilities for s , and the corresponding $q - 1$ characters have pairwise distinct restrictions to T by the previous paragraph. \square

REMARK 3.20. In Theorem 3.19(iii) (in the notation of its proof), $\text{Ind}_T^G(\mu)$ can miss some nontrivial irreducible character of G precisely when $\mu = \lambda_{s,0}|_T$ for some $0 \leq s \leq q^2 - 2$.

4. Exceptional groups of Lie type

We begin with the following observation:

LEMMA 4.1. Let \mathbf{G} be a connected reductive simply connected algebraic group in characteristic p and let $G := \mathbf{G}^{Fr}$. Let \mathbf{T} be an Fr -stable maximal torus in \mathbf{G} and let $T := \mathbf{T}^{Fr}$. Let $\chi \in \text{Irr}(G)$ lie above the irreducible character α of $Z(G)$. Suppose that every element in $T \setminus Z(G)$ is regular and that $\chi(1) \cdot |Z(G)| \geq |T|^{3/2}$. Then $\chi|_T$ contains every irreducible character of T that lies above α .

Proof. Consider any $\lambda \in \text{Irr}(T)$ lying above α and any $g \in T \setminus Z(G)$. By the assumption, $C_{\mathbf{G}}(g)$ is a torus containing \mathbf{T} , hence $C_G(g) = T$. It is well known that $Z(G) \leq T$. Now the

orthogonality relations imply that $|\chi(g)| \leq |T|^{1/2}$. It follows that

$$\begin{aligned} |T| \cdot |[\chi|_T, \lambda|_T| &= |\sum_{g \in T} \chi(g) \bar{\lambda}(g)| \geq |\sum_{g \in Z(G)} \chi(g) \bar{\lambda}(g)| - |\sum_{g \in T \setminus Z(G)} \chi(g) \bar{\lambda}(g)| \\ &\geq \chi(1) \cdot |Z(G)| - (|T| - |Z(G)|) |T|^{1/2} > \chi(1) \cdot |Z(G)| - |T|^{3/2} \geq 0. \end{aligned}$$

□

In what follows, by a *finite exceptional group of Lie type of simply connected type* we mean any *quasisimple* group $G = \mathbf{G}^{Fr}$ of type G_2 , 2G_2 , 2B_2 , 3D_4 , F_4 , 2F_4 , E_6 , 2E_6 , E_7 , or E_8 , where \mathbf{G} is simply connected. This excludes the solvable cases, as well as ${}^2G_2(3)$, $G_2(2)$, and ${}^2F_4(2)$ which are not perfect.

THEOREM 4.2. *Let G be a finite exceptional group of Lie type of simply connected type. Then G contains a cyclic maximal torus T such that the following statements hold.*

- (i) *For any $\alpha \in \text{Irr}(Z(G))$ and any non-principal $\chi \in \text{Irr}(G)$ lying above α , $\chi|_T$ contains every irreducible character of T that lies above α . In particular, if $\vartheta \in \text{Irr}(G/Z(G))$ is non-principal, then $\chi|_T$ contains 1_T .*
- (ii) *There is some $s \in T$ such that $C_{G/Z(G)}(sZ(G)) = T/Z(G)$; in particular, $T/Z(G)$ is self-centralizing in $G/Z(G)$.*

Proof. 1) First we consider the case $G = E_6(q)_{sc}$ and choose $T = \mathbf{G}^F$ to be a maximal torus of order $\Phi_9(q)$, where $\Phi_m(q)$ denotes the m^{th} cyclotomic polynomial in q . Observe that any $g \in T \setminus Z(G)$ is regular. Indeed, since \mathbf{G} is simply connected, $C_{\mathbf{G}}(g)$ is connected, cf. [2, Theorem 3.5.6]. Write $q = p^f$, where p is a prime. Then, since $g \notin Z(G)$, $C_G(g) < G$. Furthermore, $C_G(g) \geq T$ has order divisible by a primitive prime divisor ℓ of $p^{9f} - 1$, i.e. a prime that does not divide $\prod_{i=1}^{9f-1} (p^i - 1)$, cf. [33]. Using the description of the centralizers of semisimple elements in \mathbf{G} given in [4] (and noting that centralizers of type $SL_3(q^3)$ do not occur since \mathbf{G} is simply connected), we see that $C_{\mathbf{G}}(g)$ is a torus, whence g is regular and $C_G(g) = T$. Moreover, if we choose $g \in T$ to be of order ℓ , then since ℓ is coprime to $|Z(G)| = \gcd(3, q - 1)$, we obtain that

$$C_{G/Z(G)}(gZ(G)) = C_G(g)/Z(G) = T/Z(G).$$

Furthermore, $\chi(1) > |T|^{3/2}$ for any non-trivial $\chi \in \text{Irr}(G)$ has degree by the Landazuri-Seitz-Zalesskii bounds [19], [26] (and their improvements as recorded in [30]). The assertion follows by Lemma 4.1.

The same argument applies to the case $G = {}^2E_6(q)_{sc}$ if we choose $|T| = \Phi_{18}(q)$ (note that centralizers of type $SU_3(q^3)$ do not occur since \mathbf{G} is simply connected, cf. [5].) If $G = F_4(q)$, or $E_8(q)$, then, similarly, we choose T of order $\Phi_{12}(q)$ or $\Phi_{30}(q)$, respectively, and argue as above using [4] in the case of $F_4(q)$ and [20] in the case of $E_8(q)$ (which classifies maximal subgroups of maximal rank in $E_8(q)$).

Suppose $G = {}^2F_4(q)$ with $q = 2^{2a+1} \geq 8$, or ${}^2G_2(q)$ with $q = 3^{2a+1} \geq 27$. Then we choose T of order $q^2 + q + 1 + (q + 1)\sqrt{2q}$, respectively $q + \sqrt{3q} + 1$, and argue as above using [5].

2) Suppose $G = G_2(q)$ with $q \geq 3$ and $q \not\equiv -1 \pmod{3}$. Then we can choose T of order $\Phi_6(q)$ and argue as above (noting that centralizers of type $SU_3(q)$ do not occur under our hypothesis on q). Next suppose $G = G_2(q)$ with $q \geq 5$ and $q \equiv -1 \pmod{3}$. Choosing T of order $\Phi_3(q)$ and arguing as above (noting that centralizers of type $SL_3(q)$ do not occur under our hypothesis on q), we see that any element $g \in T \setminus \{1\}$ is regular with $C_G(g) = T$, and so again we are done by Lemma 4.1 if $\chi(1) \geq (q^2 + q + 1)^{3/2}$. If $\chi(1) < (q^2 + q + 1)^{3/2}$, then in

fact $\chi(1) = q^3 + 1$ and $|\chi(g)| = 1$ for all $1 \neq g \in T$ (see e.g. [16, Anhang B]), hence the proof of Lemma 4.1 yields the claim.

Consider the case $G = {}^3D_4(q)$. Choosing T of order $\Phi_{12}(q)$ and arguing as above using [5], we see that any element $g \in T \setminus \{1\}$ is regular with $C_G(g) = T$, and so again we apply Lemma 4.1, provided that $\chi(1) \geq (q^4 - q^2 + 1)^{3/2}$. If $\chi(1) < (q^4 - q^2 + 1)^{3/2}$, then by [6] in fact $\chi(1) = q(q^4 - q^2 + 1)$. Now if $1 \neq g \in T$, then g is r -singular for some prime r dividing $\chi(1)$ and χ has r -defect 0, whence $\chi(g) = 0$. Thus the claim follows from the proof of Lemma 4.1.

Suppose $G = {}^2B_2(q)$ with $q = 2^{2a+1} \geq 8$, then we choose T of order $q + \sqrt{2q} + 1$. Then for any $1 \neq g \in T$ and any $1_G \neq \chi \in \text{Irr}(G)$, $C_G(g) = T$, $|\chi(g)| \leq 1$ but $\chi(1) > |T|$ (see e.g. [1]). Hence the proof of Lemma 4.1 yields the claim.

3) Finally, we consider the case $G = E_7(q)_{sc}$. Note that $G/Z(G)$ contains a subgroup $S \cong PSL_2(q^7)$, cf. [20, Table 5.1]. We claim that $E_7(q)_{sc}$ contains a subgroup $L \cong SL_2(q^7)$. For this it suffices to consider q odd. We can view $(E_7)_{sc}$ as a component of the centralizer of an involution in E_8 . The subsystem A_1^7 of E_7 is described for example in Lemma 2.1 of [21]. It is shown there that the subgroup H with $H^0 = A_1^7$ acts irreducibly on the 56-dimensional module of E_7 , the central involution acting there as $-\text{Id}$. It follows that $SL_2(q^7)$ is a subgroup of $E_7(q)_{sc}$, as claimed. Notice that G contains a maximal torus T of order $q^7 - 1$ which contains a Sylow ℓ -subgroup for some primitive prime divisor ℓ of $q^7 - 1$. We may assume that an element $s \in T$ of order ℓ is contained in L . On the one hand, using [10] we see that s is regular and $C_G(s) = T$. Since $\ell > 2 \geq |Z(G)| = \gcd(2, q - 1)$, we have

$$C_{G/Z(G)}(sZ(G)) = C_G(s)/Z(G) = T/Z(G).$$

On the other hand, $|C_L(g)| = q^7 - 1$. Thus T can be embedded in L as a maximal torus. Now if $\alpha = 1_{Z(G)}$, then $\chi|_L$ contains a faithful irreducible character ς of $L/Z(G)$. If $Z(G) \neq 1$ and $\alpha \neq 1_{Z(G)}$, then $\chi|_L$ contains a faithful irreducible character ς of L . In either case, by Proposition 3.3, we conclude that $\varsigma|_T$ contains all irreducible characters of T lying above α . \square

REMARK 4.3. In the case where $G \in \{E_8(q), F_4(q), {}^2F_4(q), G_2(q)\}$, there is an alternative (and perhaps more conceptual) way to prove the result. As is observed in [32], see also [13], for every torus T of G and every complex irreducible representation ϕ of G the restriction $\phi|_T$ contains 1_T . This follows by taking a reduction of ϕ modulo p (in the sense of Brauer) and observation that every restricted irreducible p -modular representation of \mathbf{G} has weight 0. Note, however, that this method cannot be used when α is non-trivial.

In the next statement, we also include $G_2(2)'$, ${}^2G_2(3)'$, and ${}^2F_4(2)'$ among the simple exceptional groups of Lie type. In the last two cases, $G = {}^2G_2(3)'$ and $G = {}^2F_4(2)'$, by a Steinberg character of G we understand any of the irreducible constituents of the groups $G = {}^2G_2(3)$ and $G = {}^2F_4(2)$, of degree and 9 and 2048, respectively.

COROLLARY 4.4. *Theorems 1.1, 1.2, and 1.7 hold for simple exceptional groups of Lie type, except for $G_2(2)' \cong SU_3(3)$.*

Proof. By Theorem 4.2 and Corollary 5.2 (below), this is true unless G is one of the following groups ${}^2G_2(3)'$, $G_2(2)'$, ${}^2F_4(2)'$. In the first case, we have an exception. In the second case, ${}^2G_2(3)' \cong SL_2(8)$, and Theorem 1.1 follows by previous result for $SL_2(8)$, and Theorem 1.2 holds by direct computation. In the third case, 1.1 holds by choosing $T = C_G(s)$ of order 13, and Theorem 1.2 holds by direct computation (or by applying [15, Lemma 3.1]). \square

Proof of Theorem 1.6. Obviously, $\Delta_G \subseteq \Delta_C$. As the group algebra $\mathbb{C}G$ is a direct sum of simple rings, so are Δ_G and Δ_C , and hence Δ_G is an ideal of Δ_C . There is a natural bijection between simple rings in question and $\text{Irr } G$. It follows the direct summands of Δ_G , resp., Δ_C corresponds to the irreducible representations of G that do not occur in Π_G resp., Π_C . By Theorem 1.7, $\Delta_C = 0$, unless $G \cong PSU_n(q)$ with $(2(q+1), n) = 1$, so the assertion follows in the non-exceptional case. In the exceptional case there is a single representation of G not occurring in Π_C . Therefore, Δ_C is a simple ring, so either $\Delta_G = 0$ or $\Delta_G = \Delta_C$. However, $\Delta_G \neq 0$ by [15, Theorem 1.1]. So the result follows.

5. The square of the Steinberg character

Next we turn our attention to the square of the Steinberg character St of a simple group G of Lie type. Such a group can be obtained from a simple simply connected algebraic group \mathbf{G} as $L/Z(L)$, where $L = \mathbf{G}^{Fr}$ is the fixed point subgroup of a Frobenius endomorphism $Fr : \mathbf{G} \rightarrow \mathbf{G}$ and L is quasisimple. Let \mathbf{T} be an Fr -stable maximal torus in \mathbf{G} . Then $T = \mathbf{T} \cap L$ is called a maximal torus of L . Set $W(T) = (N_{\mathbf{G}}(\mathbf{T})/\mathbf{T})^{Fr}$.

LEMMA 5.1. *Let χ be an irreducible character of $L = \mathbf{G}^{Fr}$. Then*

$$[\chi \cdot \text{St}, \text{St}]_G = \sum_{(\mathbf{T})} \frac{[\chi|_T, 1_T]_T}{|W(T)|},$$

where the sum ranges over representatives of the G -conjugacy classes of Fr -stable maximal tori \mathbf{T} of \mathbf{G} and $T = \mathbf{T}^{Fr}$.

Proof. By [3, 7.15.2], $\text{St} \cdot \text{St} = \sum_{(\mathbf{T})} \frac{1}{|W(T)|} 1_T^G$, where the sum ranges as above. Note that $[\chi, 1_T^G]_G = [\chi|_T, 1_T]_T$ by the Frobenius reciprocity. This implies the lemma. \square

COROLLARY 5.2. *Let G be a simple group of Lie type, let St be the Steinberg character of G , and let χ be any irreducible character of G . Then χ is a constituent of the tensor square St^2 if and only if for some maximal torus T of G , the restriction of χ to T involves 1_T .*

Proof. Again view G as $L/Z(L)$ for some $L = \mathbf{G}^F$ as above. Since the Steinberg character of L is trivial at $Z(L)$, we may replace G by L and χ by any irreducible character of L trivial at $Z(L)$. As St is self-dual, we have $[\chi, \text{St} \cdot \text{St}]_G = [\chi \cdot \text{St}, \text{St}]_G$. Now we apply Lemma 5.1 and note that all terms on the right hand side of the formula are non-negative. Therefore, $[\chi \cdot \text{St}, \text{St}]_G > 0$ whenever there exists a maximal torus T of L such that $[\chi|_T, 1_T]_T > 0$. \square

The assertion of Corollary 5.2 holds also for the groups $Sp_4(2)', 2G_2(3)', 2F_4(2)'$, with the Steinberg character and torus suitably defined, but not for the group $G_2(2)'$, where the irreducible characters of degree 32 are not real.

One might think that in the exceptional case $G = SU_n(q)$ with $(n, 2(q+1)) = 1$ in Theorem 1.2, one could try to replace St^2 by τ^2 or $\tau\bar{\tau}$ for some other $\tau \in \text{Irr } G$ so that the resulting character would include every $\chi \in \text{Irr}(G)$ as an irreducible character. It is however not the case.

LEMMA 5.3. *Let $G = SU_n(q)$, $n \geq 3$, and $(n, 2(q+1)) = 1$. Let τ be an arbitrary complex irreducible character of G and let ϕ_{\min} be the irreducible non-trivial character of minimum*

degree $(q^n - q)/(q + 1)$. Then no $\varrho \in \{\tau^2, \tau\bar{\tau}\}$ can contain both 1_G and ϕ_{\min} as irreducible constituents.

Proof. Note that the permutation character π of the conjugation action is $\sum_{\chi \in \text{Irr}(G)} \chi \bar{\chi}$ and it does not contain ϕ_{\min} by Theorem 1.1. Hence the assertion follows if $\varrho = \tau\bar{\tau}$, or if $\varrho = \tau^2$ and $\tau = \bar{\tau}$. If $\tau \neq \bar{\tau}$, then $[1_G, \tau^2]_G = [\bar{\tau}, \tau]_G = 0$. \square

Proof of Theorem 1.2. This follows from Theorem 1.1 and Corollary 5.2.

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